Performance Evaluation and Networks

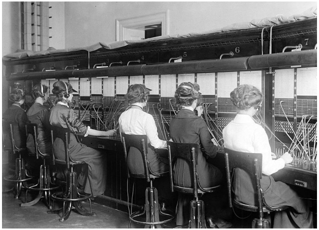
Markov Chains (MC)
when time is continuous

Poissonian traffic

Poissonian traffic: model of traffic where time is continuous and the inter-arrival times between packets are i.i.d. random variables of exponential law $Exp(\lambda)$, $\lambda > 0$.

History: used by Erlang to model incoming phone calls at call centers of the Copenhagen Telephone Company (1909)

$\rightarrow$ optimized dimensioning of the call centers, mainly number of human operators and number of cord boards at that time.



Telephone Operators, Washington DC, USA, by Harris & Ewing, 1915

Counting process: definition

Framework: continuous time stochastic processes

Definition (counting process)

Three different descriptions of a counting process:

- *Arrival times* $(T_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}_+$: $T_0 = 0$, *if* $m < n$, *then* $T_m \leq T_n$ *a.s.*
- *Inter-arrival times* $(I_n)_{n\in\mathbb{N}^*}$ *in* $\mathbb{R}_+$
- *Counter* $(N_t)_{t\in\mathbb{R}_+}$ *in* $\mathbb{N}\cup\{\infty\}$: *if* $s < t$, *then* $N_s \leq N_t$ *a.s.*

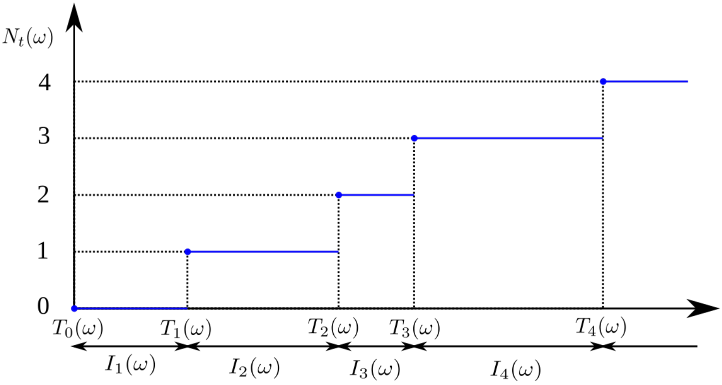
equivalent via the next relations:

- $\forall n \geq 1,\ T_n = I_1 + \cdots + I_n$
- $\forall n \geq 1,\ I_n = T_n - T_{n-1}$
- $\forall t \geq 0,\ N_t = \sup\{n \in \mathbb{N} | T_n \leq t\}$
- $\forall n \geq 0,\ T_n = \inf\{t \in \mathbb{R}_+ | N_t \geq n\}$

Poisson process: definition

Definition (poisson process)

Let $\lambda > 0$*, a* poisson process *of intensity* λ *is a counting process where inter-arrivals* $(I_n)_{n \in \mathbb{N}^*}$ *are i.i.d. exponential laws of parameter* λ*.*



Poisson process: sum of exponential clocks

Proposition (0/1 law of cumulative exponential clocks)

Let $(E_n)_{n\geq 1}$ *independent r.v. of respective laws* $Exp(\lambda_n)$ *with* $\lambda_n > 0$.

$$\text{Then} \begin{cases} \mathbb{P}\left(\sum_{n=1}^{\infty} E_n = \infty\right) = 1 & \text{if } \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty \\ \mathbb{P}\left(\sum_{n=1}^{\infty} E_n = \infty\right) = 0 & \text{if } \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty \end{cases}$$

Corollary

Given a poisson process, $\mathbb{P}(T_n \to \infty) = 1$ *and thus* N_t *is finite a.s.*

Poisson process: sum of exponential clocks

Proof of 0/1 law of cumulative exponential clocks:

Random variable $T_\infty = \sum_{n=1}^{\infty} E_n$ has values in $\mathbb{R}_+ \cup \{\infty\}$

Use monotone convergence of $\mathbb{E}$:

$\mathbb{E}(T_\infty) = \sum_{n=1}^{\infty} \mathbb{E}(E_n) = \sum_{n=1}^{\infty} 1/\lambda_n \in \mathbb{R}_+ \cup \{\infty\}$

Thus if $\mathbb{E}(T_\infty) = \sum_{n=1}^{\infty} 1/\lambda_n < \infty$, of course $\mathbb{P}(T_\infty = \infty) = 0$

Using monotone convergence of $\mathbb{E}$ and independence,

$$\mathbb{E}(e^{-T_\infty}) = \prod_{n=1}^{\infty} \mathbb{E}(e^{-E_n}) = \prod_{n=1}^{\infty} (1 + \frac{1}{\lambda_n})^{-1}$$

since $\mathbb{E}(e^{-E_n}) = \int_0^\infty \lambda_n e^{-\lambda_n x} e^{-x} dx = \lambda_n/(1+\lambda_n)$

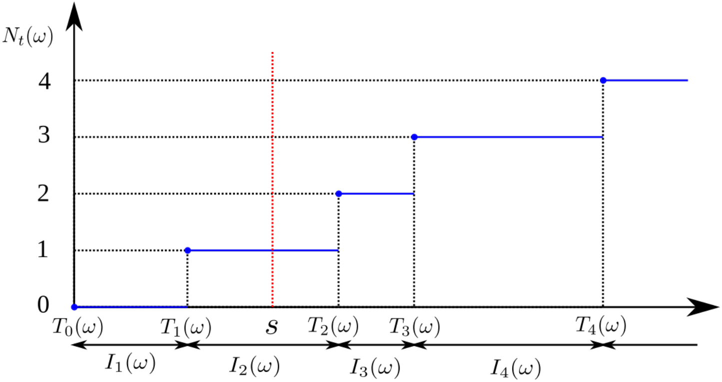
Remark that $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$ iff $\prod_{n=1}^{\infty}(1+1/\lambda_n)^{-1} = 0$, thanks to a classical lemma: let (a_n) non negative reals, $\prod_{n=1}^{\infty}(1+a_n)$ converges iff $\sum_{n=1}^{\infty} a_n$ converges.

Thus if $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$, then $\mathbb{E}(e^{-T_\infty}) = 0$ and $\mathbb{P}(T_\infty = \infty) = 1$.

Poisson process: markov property

Proposition (poisson processes are markovian)

Let $(N_t)_{t\in\mathbb{R}_+}$ *a poisson process of intensity* λ*. Then for all* $s \geq 0$, $(N_{t+s} - N_s)_{t\in\mathbb{R}_+}$ *is a poisson process of intensity* λ*, indep of* $(N_r)_{0\leq r\leq s}$

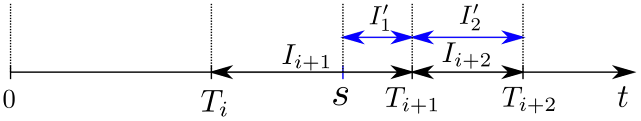


Poisson process: markov property

Proof: at time s (the present), suppose that the counter is at state $i \in \mathbb{N}$, consider $\{N_s = i\} = \{T_i \leq s < T_{i+1}\} = \{T_i \leq s\} \cap \{s < T_i + I_i\}$
Inter-arrivals (I'_n) for $(N_{t+s} - N_s)_{t \in \mathbb{R}_+}$ satisfy:

$$I'_1 = I_{i+1} - (s - T_i) \text{ and } I'_n = I_{i+n} \text{ for } n \geq 2$$

Condition on $\{N_s = i\}$ and $I_1, \dots, I_i$ (i.e. for any set of events $\{I_k \in B_k\}$ where B_k borelian of $\mathbb{R}$): $I'_1 \sim Exp(\lambda)$ due to the memoryless property of $I_{i+1} \sim Exp(\lambda)$ (i.e. $\mathbb{P}(I_{i+1} > v + u | I_{i+1} > v) = \mathbb{P}(I_{i+1} > u)$) and is independent of $(I_n)_{n \neq i+1}$ as I_{i+1}. Moreover, for all $n \geq 2$, $I'_n = I_{i+n} \sim Exp(\lambda)$ are independent, and independent of $I_1, \dots, I_i$.
To sum up, conditional on $\{N_s = i\}$, (I'_n) are i.i.d $\sim Exp(\lambda)$ and independent of $I_1, \dots, I_i$, thus independent of $(N_r)_{0 \leq r \leq s}$.



Poisson process: Gamma law

Proposition (law of poisson arrival times)

For a poisson process of intensity λ, the arrival time T_n follows the Gamma distribution $\Gamma(n, \lambda)$ of density $f(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \mathbb{1}_{\mathbb{R}_+}(t)$

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Proof: by induction on n. True for $n = 1$: clearly $\Gamma(1,\lambda) = Exp(\lambda)$. Suppose $T_n \sim \Gamma(n,\lambda)$. Then $T_{n+1} = T_n + I_{n+1}$ with $I_{n+1} \sim Exp(\lambda)$ indep of T_n. Thus the density of T_{n+1} is the convolution:

$$
\begin{aligned}
f(t) &= \int_{u+v=t} \frac{1}{(n-1)!} \lambda^n u^{n-1} e^{-\lambda u} \mathbb{1}_{\mathbb{R}_+}(u) \lambda e^{-\lambda v} \mathbb{1}_{\mathbb{R}_+}(v) du \\
&= \frac{\lambda^{n+1} e^{-\lambda t}}{(n-1)!} \int_{u=0}^{t} u^{n-1} du \\
&= \frac{\lambda^{n+1} e^{-\lambda t}}{(n-1)!} \frac{t^n}{n}
\end{aligned}
$$

Poisson process: Poisson law

Proposition (law of poisson counters)

For a poisson process of intensity λ, the counter N_t follows the Poisson distribution $\mathscr{P}(\lambda t)$.

Proof: $\{N_t = n\} = \{T_n \leq t < T_{n+1}\}$, hence:

$$\begin{aligned}
\mathbb{P}(N_t = n) &= \mathbb{P}(T_n \leq t) - \mathbb{P}(T_{n+1} \leq t) \\
&= \int_{x=0}^{t} \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x} dx - \int_{x=0}^{t} \frac{\lambda^{n+1} x^n}{n!} e^{-\lambda x} dx \\
&= \int_{x=0}^{t} \frac{\lambda^n}{n!} \left[n x^{n-1} e^{-\lambda x} - \lambda x^n e^{-\lambda x} \right] dx \\
&= \frac{\lambda^n}{n!} \left[x^n e^{-\lambda x} \right]_{x=0}^{x=t} \\
&= \frac{\lambda^n}{n!} t^n e^{-\lambda t} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}
\end{aligned}$$

Poisson process: equivalent definitions

Theorem (characterizations of poisson processes)

Let $\lambda > 0$ and a counting process, the next statements are equivalent:

1. *Inter-arrivals: I_n i.i.d. $\sim Exp(\lambda)$*
2. *Macro-increments: $\forall 0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$, the increments $N_{t_n} - N_{t_{n-1}}, \ldots, N_{t_1} - N_{t_0}$ are indep and have respective laws $\mathscr{P}(\lambda(t_n - t_{n-1})), \ldots, \mathscr{P}(\lambda(t_1 - t_0))$*
3. *Micro-increments: increments are indep and uniformly in t, $\mathbb{P}(N_{t+\varepsilon} - N_t = 0) = 1 - \lambda\varepsilon + o(\varepsilon)$, $\mathbb{P}(N_{t+\varepsilon} - N_t = 1) = \lambda\varepsilon + o(\varepsilon)$*

Vocabulary: "uniformly in t" = $o(\varepsilon)$ doesn't depend on t

Proof (1) $\Rightarrow$ (2): markov + poisson distribution for counters
Proof (2) $\Rightarrow$ (1): (3) defines a unique process, thus same as (1)
Proof (2) $\Rightarrow$ (3): $\mathbb{P}(N_{t+\varepsilon} - N_t = 0) = e^{-\lambda\varepsilon}$, $\mathbb{P}(N_{t+\varepsilon} - N_t = 1) = e^{-\lambda\varepsilon}\lambda\varepsilon$

Poisson process: equivalent definitions

Proof (3) ⇒ (2): for all $n \geq 2$, $\mathbb{P}(N_{t+\varepsilon} - N_t = n) = o(\varepsilon)$ uniformly in t.
Set $p_n(t) = \mathbb{P}(N_t = n)$. Then for all $n \geq 1$,

$$\begin{aligned} p_n(t+\varepsilon) = \mathbb{P}(N_{t+\varepsilon}) &= \sum_{i=0}^{n} \mathbb{P}(N_{t+\varepsilon} - N_t = i)\mathbb{P}(N_t = n-i) \\ &= (1 - \lambda\varepsilon + o(\varepsilon))p_n(t) + (\lambda\varepsilon + o(\varepsilon))p_{n-1}(\varepsilon) + o(\varepsilon) \end{aligned}$$

Thus $\frac{p_n(t+\varepsilon)-p_n(t)}{\varepsilon} = -\lambda p_n(t) + \lambda p_{n-1}(t) + o(1)$
Since this estimate is uniform in t, we can change t into $t-\varepsilon$,
yielding for all $t \geq \varepsilon$, $\frac{p_n(t)-p_n(t-\varepsilon)}{\varepsilon} = -\lambda p_n(t-\varepsilon) + \lambda p_{n-1}(t-\varepsilon) + o(1)$
Letting $\varepsilon \downarrow 0$, $p_n(t)$ continuous and differentiable such that:
$p_n'(t) = -\lambda p_n(t) + \lambda p_{n-1}(t)$. In the same way, $p_0'(t) = -\lambda p_0(t)$
Initial conditions: $N_0 = 0$ a.s. thus $p_0(0) = 1$ and $p_n(0) = 0$ for $n \geq 1$
Solve (by induction on n): $p_n(t) = e^{-nt}\frac{(\lambda t)^n}{n!}$
Thus $N_t \sim \mathscr{P}(\lambda t)$. Then note that for any $s \geq 0$, $(N_{t+s} - N_s)_{t\in\mathbb{R}_+}$ also satisfies (3), yielding $N_{t+s} - N_s \sim \mathscr{P}(\lambda t)$. Independence of increments is present both in (3) and (2).

Poisson process: transformations

Corollary (superposition of indep poisson processes)

If two poisson processes of intensity λ (resp λ') and counter $(N_t)_{t\in\mathbb{R}_+}$ (resp. $(N'_t)_{t\in\mathbb{R}_+}$) are independent, then $(N_t+N'_t)_{t\in\mathbb{R}_+}$ is a poisson process of intensity $\lambda+\lambda'$.

Sketch of proof: use micro-increments characterization (3)

$$\mathbb{P}(N_t+N'_t=0)=\mathbb{P}(N_t=0,N'_t=0)\overset{\text{indep}}{=}\mathbb{P}(N_t=0)\mathbb{P}(N'_t=0)$$

$$\mathbb{P}(N_t+N'_t=1)\overset{\text{indep}}{=}\mathbb{P}(N_t=0)\mathbb{P}(N'_t=1)+\mathbb{P}(N_t=1)\mathbb{P}(N'_t=0)$$

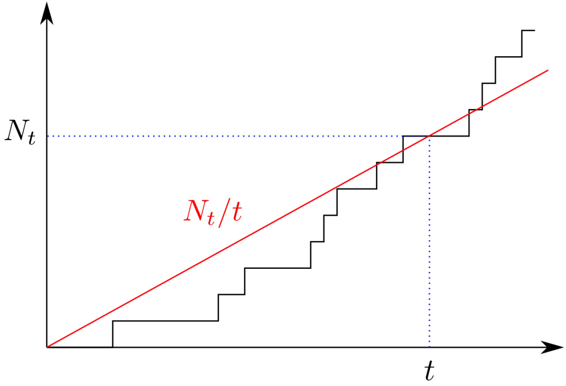
Proposition (thinning of a poisson process)

Given a poisson process and $(B_n)_{n\in\mathbb{N}^*}$ i.i.d. $\sim\mathscr{B}(p)$ indep of (T_n). Then the two counting processes defined by arrival times $\{T_n \text{ s.t. } B_n=1\}$, resp. $\{T_n \text{ s.t. } B_n=0\}$, are indep and of respective intensities λp and $\lambda(1-p)$.

Poisson process: limit theorems

Definition: arrival rate up to time $t \stackrel{\text{def}}{=} \frac{N_t}{t}$

Question: evolution of $\frac{N_t}{t}$ when $t \to +\infty$?



Poisson process: limit theorems

Theorem (LLN and CLT for poisson processes)

Given a poisson process of intensity $\lambda > 0$, then

$$\frac{N_t}{t} \xrightarrow[t\to+\infty]{a.s.} \lambda \quad \textit{and} \quad \sqrt{t}\left(\frac{N_t}{t} - \lambda\right) \xrightarrow[t\to+\infty]{law} \mathcal{N}(0,\lambda)$$

Proof of LLN: first note that $N_{T_n} = n$.
LLN for T_n: $\frac{T_n}{n} = \frac{I_1+\cdots+I_n}{n} \xrightarrow[n\to+\infty]{\text{a.s.}} \frac{1}{\lambda}$ since (I_n) i.i.d. $\sim Exp(\lambda)$.
Let $t \in \mathbb{R}_+$, $\exists n \in \mathbb{N}$, $T_n \leq t < T_{n+1}$, which also satisfies $t \to +\infty$ iff $T_n \to +\infty$ iff $n \to +\infty$ (since $T_n \nearrow +\infty$). Bound and let $t \to +\infty$:

$$\underbrace{\frac{n}{n+1}\frac{n+1}{T_{n+1}}}_{\xrightarrow[\text{a.s.}]{}\lambda} = \frac{n}{T_{n+1}} = \frac{N_{T_n}}{T_{n+1}} \leq \frac{N_t}{t} \leq \frac{N_{T_{n+1}}}{T_n} = \frac{n+1}{T_n} = \underbrace{\frac{n}{T_n}\frac{n+1}{n}}_{\xrightarrow[\text{a.s.}]{}\lambda}$$

Continuous time MC: definition

Definition (continous time MC)

$(X_t)_{t\in\mathbb{R}_+}$ *markovian* if $\forall n \in \mathbb{N}$, $\forall t_0 < \cdots < t_n < t_{n+1}$,
$\forall x_0, \ldots, x_n, x_{n+1} \in E$,

$$\mathbb{P}(X_{t_{n+1}} = x_{n+1} | X_{t_n} = x_n, \ldots, X_{t_0} = x_0) = \mathbb{P}(X_{t_{n+1}} = x_{n+1} | X_{t_n} = x_n)$$

Definition (continous time HMC)

$(X_t)_{t\in\mathbb{R}_+}$ *markovian and homogeneous* if $\forall n \in \mathbb{N}$,
$\forall t_0 < \cdots < t_n < t_{n+1}$, $\forall x_0, \ldots, x_n, x_{n+1} \in E$,

$$\mathbb{P}(X_{t_{n+1}} = x_{n+1} | X_{t_n} = x_n, \ldots, X_{t_0} = x_0) = \mathbb{P}(X_{t_{n+1}-t_n} = x_{n+1} | X_0 = x_n)$$

Notation for HMC: $p_{ij}(t) = \mathbb{P}(X_t = j | X_0 = i)$ and $P(t) = (p_{ij}(t))_{i,j\in E}$.
Discrete time HMC: $P(n) = P^n$ where
$P = P(1) = (\mathbb{P}(X_1 = j | X_0 = i))_{i,j\in E}$
Continuous time HMC: $P(t)$ when $t \to 0$?

Continuous time HMC: semigroup structure

Proposition (semigroup of transition matrices)

Let $(X_t)_{t\in\mathbb{R}_+}$ HMC on E, then its transition matrices $P(t)$, $t \in \mathbb{R}_+$, form a sub-semigroup of stochastic matrices, called transition semigroup*:*

1. $P(t)$ *is a stochastic matrix* $\forall t \geq 0$
2. $P(0) = Id_E$ *the identity matrix on* E
3. $P(s+t) = P(s)P(t), \forall s, t \geq 0$

Remark: semigroup structure = (2) + (3)

Proof: (3) = Chapman-Kolmogorov equations still valid at continuous times

Continuous time HMC: semigroup structure

Definition (continuous semigroup)

A semigroup of matrices $P(t)$, $t \in \mathbb{R}_+$, *is continuous if* $\lim_{\varepsilon \downarrow 0} P(\varepsilon) = P(0) = Id_E$ *(pointwise convergence).*

Theorem (local characteristics of continuous semigroups)

Let $P(t)$, $t \in \mathbb{R}_+$, *be a continuous transition semigroup on a countable space* E. *Then*

1. $\forall i \in E, \exists q_i \stackrel{def}{=} \lim_{\varepsilon \downarrow 0} \frac{1-p_{ii}(\varepsilon)}{\varepsilon} \in \mathbb{R}_+ \cup \{\infty\}$
2. $\forall i, j \in E, \exists q_{ij} \stackrel{def}{=} \lim_{\varepsilon \downarrow 0} \frac{p_{ij}(\varepsilon)}{\varepsilon} \in \mathbb{R}_+$

Continuous time HMC: infinitesimal generator

Definition (infinitesimal generator)

Let $P(t)$, $t \in \mathbb{R}_+$, be a continuous transition semigroup on a countable space E, its infinitesimal generator is $Q = (q_{ij})_{i,j \in E}$ with $q_{ii} = -q_i$.

Compact notation: $Q = \lim_{\varepsilon \downarrow 0} \frac{P(\varepsilon) - P(0)}{\varepsilon}$

Continuous time HMC: stability & conservation

Definition (stability & conservation)

Let Q the infinitesimal generator of a continuous transition semigroup, this semigroup is called

1. *stable if* $\forall i \in E$, $q_i < \infty$
2. *conservative if* $\forall i \in E$, $q_i = \sum_{j \neq i} q_{ij}$

Remark: always true when E finite

Use: $\sum_{j \in E} p_{ij}(\varepsilon) = 1$ implies $\frac{1 - p_{ii}(\varepsilon)}{\varepsilon} = \sum_{j \neq i} \frac{p_{ij}(\varepsilon)}{\varepsilon}$ and $q_i = \lim_{\varepsilon \downarrow 0} \sum_{j \neq i} \frac{p_{ij}(\varepsilon)}{\varepsilon}$

Thus stability & conservation ensures that we can invert lim and $\sum$:

$$\lim_{\varepsilon \downarrow 0} \sum_{j \neq i} \frac{p_{ij}(\varepsilon)}{\varepsilon} = \sum_{j \neq i} \lim_{\varepsilon \downarrow 0} \frac{p_{ij}(\varepsilon)}{\varepsilon}$$

Continuous time HMC: regular jump HMC

Definition (regular jump process)

A stochastic process $(X_t)_{t\in\mathbb{R}_+}$ *is a jump process if for almost all* $\omega \in \Omega$, $\forall t \geq 0, \exists \varepsilon(t,\omega) > 0$ *such that* $X_{t+s}(\omega) = X_t(\omega), \forall s \in [t, t+\varepsilon(t,\omega)[$.
It is called regular if, in addition, for almost all $\omega \in \Omega$, *the set* $D(\omega)$ *of discontinuities of* $t \mapsto X_t(\omega)$ *satisfies* $\forall t \geq 0, |D(\omega) \cap [0,t]| < \infty$.

Interpretation: trajectories with staircase shape (jump process) and no accumulation point of discontinuities (regular).
Regular jump HMC: HMC which is also a regular jump process

Theorem (regular jump HMC ⇒ stable & conservative)

A regular jump HMC is stable and conservative.

Jump processes: transition times

Proposition (transition times & embedded process)

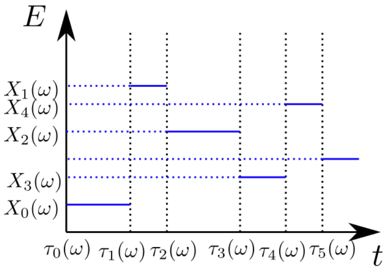
For a jump process $(X_t)_{t\in\mathbb{R}_+}$ *(not necessarily regular), there exists a sequence of time r.v.* $(\tau_n)_{n\geq 0}$ *and state r.v.* $(X_n)_{n\geq 0}$ *such that:*
$\tau_0 = 0 < \tau_1 < \tau_2 < \cdots$ *and* $\forall \tau_n \leq t < \tau_{n+1}, X_t = X_n$.
We call (τ_n) *the transition times and* (X_n) *the embedded process.*

Interpretation: X_n = state after n jumps, where we stay during time interval $[\tau_n, \tau_{n+1}[$ (holding period)

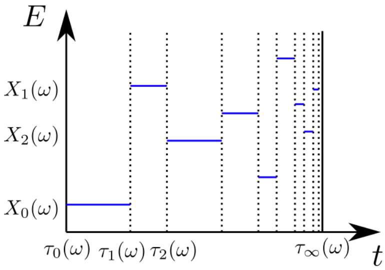
Proposition (explosion time)

The explosion time is $\tau_\infty = \lim_n \tau_n$. *If the jump process is regular, then* $\tau_\infty = \infty$ *a.s.*

Jump processes: regular vs non-regular



Trajectory of a
regular jump process



Trajectory of a
non-regular jump process

Kolmogorov's differential systems: finite state space

Proposition (continuous time HMC over finite state space)

Let $(X_t)_{t\in\mathbb{R}_+}$ HMC over finite E, then it is a regular jump HMC, with continuous, conservative and stable transition semigroup. Moreover, let Q its infinitesimal generator,

$$\frac{d}{dt}P(t) = P(t)Q = P(t)Q \text{ (Kolomogorov's differential systems)}$$

With initial condition $P(0) = I$, its unique solution is $P(t) = exp(tQ)$.

Reminder: $exp(M) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{M^n}{n!}$ well defined for $M \in \mathcal{M}_{|E|}(\mathbb{R})$

Proof: from $P(t+\varepsilon) = P(t)P(\varepsilon) = P(\varepsilon)P(t)$, transform into

$$\frac{P(t+\varepsilon)-P(t)}{\varepsilon} = P(t)\frac{P(\varepsilon)-I}{\varepsilon} = \frac{P(\varepsilon)-I}{\varepsilon}P(t)$$

Then $\varepsilon \downarrow 0$ with no issue to invert lim and $\sum$ (finite sums)

Kolmogorov's differential systems: general case

Theorem (backward Kolmogorov differential system)

Let $(X_t)_{t\in\mathbb{R}_+}$ HMC over E with continuous transition semigroup and infinitesimal generator Q, if it is conservative and stable, Kolmogorov's backward differential system holds:

$$\frac{d}{dt}P(t) = QP(t)$$

Theorem (forward Kolmogorov differential system)

Let $(X_t)_{t\in\mathbb{R}_+}$ HMC over E with continuous transition semigroup and infinitesimal generator Q, if it is conservative and stable and moreover $\forall\, i E$, $\forall\, t \geq 0$, $\sum_{j\in E} p_{ij}(t) q_j < \infty$, Kolmogorov's forward differential system holds:

$$\frac{d}{dt}P(t) = P(t)Q$$

Evolution of laws: global balance

Vector notation of the law $\nu(t)$ of r.v. X_t with values in E:

$$\nu(t) = (\nu_i(t))_{i \in E} \text{ line vector with } \nu_i(t) \stackrel{\text{def}}{=} \mathbb{P}(X_t = i)$$

Proposition (evolution of laws)

Let $(X_t)_{t \in \mathbb{R}_+}$ HMC over E with transition semigroup $P(t)$, $t \in \mathbb{R}_+$, then $\forall s, t \geq 0, \nu(s+t) = \nu(s)P(t)$

Theorem (global balance)

Let $(X_t)_{t \in \mathbb{R}_+}$ HMC over E with continuous semigroup and infinitesimal generator Q, if it is conservative and stable and if $\nu(t)$ distribution over E satisfies $\forall t \geq 0, \sum_{i \in E} q_i \nu_i(t) < \infty$, then Kolmogorov's global differential system holds:

$$\frac{d}{dt}\nu(t) = \nu(t)Q$$

Invariant distribution: definition

Definition (Invariant/stationnary distribution)

Invariant distribution for transition semigroup $P(t)$, $t \in \mathbb{R}_+$: *probability distribution* $\pi = (\pi_i)_{i \in E} \in \mathbb{R}_+^E$, *i.e.* $\sum_{i \in E} \pi_i = 1$, *such that* $\forall t \in \mathbb{R}_+, \pi P(t) = \pi$.

Remark: same as discrete time HMC, but does not directly simplifies to a unique equation like $\pi P = \pi$.

Proposition (Invariant distribution for finite state space)

Let $(X_t)_{t \in \mathbb{R}_+}$ *HMC over finite* E, *with infinitesimal generator* Q, *then* π *invariant distribution iff* $\pi Q = 0$

Sketch of proof: use Kolmogorov's differential system or the exponential form of $P(t)$

$$\pi Q = 0 \Leftrightarrow \forall n \geq 1, \pi Q^n = 0 \Leftrightarrow \pi \sum_{n=0}^{\infty} \frac{t^n}{n!} Q^n = \pi \Leftrightarrow \pi P(t) = \pi$$

Regular jumps over countable state space

Theorem (regular jump HMC)

Let $(X_t)_{t\in\mathbb{R}_+}$ regular jump HMC over countable state space E, with transition semigroup $P(t)$, $t\in\mathbb{R}_+$, then

1. *semigroup $P(t)$, $t\in\mathbb{R}_+$, is continuous, stable and conservative*
2. *Kolmogorov's backward and forward differential systems are satisfied, as well as global balance differential systems*
3. *probability distribution π on E is invariant for the HMC iff $\pi Q = 0$, where Q infinitesimal generator.*

In practice: most models from computer science or related fields (logistics, transportation, ...) using continuous time HMC have regular jumps.

Regular jump HMC: new characterization

Theorem (regenerative structure)

Let $(X_t)_{t\in\mathbb{R}_+}$ regular jump HMC over countable state space E, with infinitesimal generator Q, transition times (τ_n) and embedded process (X_n), then:

- *(X_n) discrete time HMC with transition proba $p_{ij} = \frac{q_{ij}}{q_i}$ if $q_i > 0$*
- *Given $(X_n)_{n\geq 0}$, holding times $(\tau_{n+1} - \tau_n)_{n\geq 0}$ are indep and respectively follow an exponential law of parameter q_{X_n}*

Vocabulary: embedded process of regular jump HMC also called embedded HMC

Regular jump HMC: new characterization

Theorem (concurrent exponential clocks)

Consider a jump process $(X_t)_{t\in\mathbb{R}_+}$ *such that, given* $X_t = i$, *an indep clock with exponential law of parameter* λ_{ij} *indicates the waiting time before a jump to* j *for each state* j *(by convention* $\lambda_{ij} = 0$ *if no possible jump from* i *to* j*). The jump occurs to the state with smallest waiting time and all clocks are reset. Then if* $\forall i, \lambda_i = \sum_j \lambda_{ij} < \infty$, *this process is a regular jump HMC with infinitesimal generator Q such that* $q_{ij} = \lambda_{ij}$.

Use: useful to identify continuous time HMC during modelization

Nice continuous-time HMC: summary

Up to adding assumptions (especially when the state space is countable but not finite), there exists four points of vue (which can be put into characterizations) of the nice regular jump HMC:

1. markovian description: $P(0) = Id$, $P(s+t) = P(s)P(t)$
2. infinitesimal description: $\frac{dP}{dt}(t) = QP(t) = P(t)Q$
3. discrete event system (DES) description: alternation of exponential waiting times and jumps with discrete time embedded markov chain
4. other DES description: concurrent jumps with exponential waiting times

Regular jump HMC: structure

Definition (irreducibility/recurrence/ t-positive recurrence)

- *A regular jump HMC is irreducible if its embedded HMC is irreducible.*
- *A state i is recurrent if it is recurrent in the embedded HMC*
- *A state i is t-recurrent positive if $\mathbb{E}_i[T_i] < \infty$*

Remark: t-positive recurrence may differ from positive recurrence in the embedded HMC due to holding times.

Proposition

An irreducible recurrent regular jump HMC with invariant measure ν is t-positive recurrent iff $\sum_i \nu(i) < \infty$.

Regular jump HMC: ergodicity

Definition (Ergodic continuous-time HMC)

An irreducible regular jump HMC is called ergodic if it is t-positive recurrent.

Theorem (Criterion of ergodicity)

An irreducible regular jump HMC with infinitesimal generator Q is ergodic iff there exists a probability distribution π on E such that $\pi Q = 0$ (invariant distribution). In that case, π is unique.

Regular jump HMC: asymptotic behavior

Theorem (Asymptotic behavior of transition probabilities)

Let $(X_t)_{t \in RR_+}$ *be an ergodic regular jump HMC over* E *with transition semigroup* $(P(t))_{t \in \mathbb{R}_+}$, *then:* $\forall i, j \in E$, $\lim_{t \to \infty} p_{ij}(t) = \pi_j$ *where* π *is the (unique) invariant distribution.*

Theorem (Ergodic theorem)

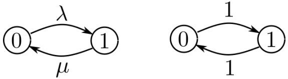
Let $(X_t)_{t \in RR_+}$ *be an ergodic regular jump HMC over* E, *and* π *its (unique) invariant distribution, then:*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X(s))ds = \sum_{i \in E} f(i)\pi_i \text{ a.s.}$$

for any initial distribution μ *and any* $f : E \to \mathbb{R}$ *such that* $\sum_{i \in E} |f(i)|\pi_i < \infty$.

Regular jump HMC: examples

Example 1: Two states switch, with respective waiting times $Exp(\lambda)$ and $Exp(\mu)$, $\lambda, \mu > 0$



Structure: irreducible, finite nb of states

Infinitesimal generator: $Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$

Invariant distribution: $\pi Q = 0 \implies \pi_0 = \frac{\mu}{\lambda+\mu}, \pi_1 = \frac{\lambda}{\lambda+\mu}$

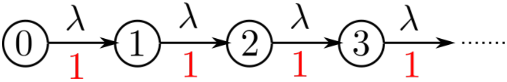
Transition matrices: $Q^2 = -(\lambda+\mu)Q \Rightarrow Q^n = (-(\lambda+\mu))^{n-1}Q$

$$P(t) = e^{tQ} = I + \sum_{n=1}^{\infty} \frac{t^n}{n!}(-(\lambda+\mu))^{n-1}Q = I - \frac{1}{\lambda+\mu}(e^{-(\lambda+\mu)t} - 1)Q$$

$$= \frac{1}{\lambda+\mu}\begin{pmatrix} \mu + \lambda e^{-(\lambda+\mu)t} & \lambda - \lambda e^{-(\lambda+\mu)t} \\ \mu - \mu e^{-(\lambda+\mu)t} & \lambda + \mu e^{-(\lambda+\mu)t} \end{pmatrix}$$

Regular jump HMC: examples

Example 2: Poisson processes of intensity $\lambda > 0$



Structure: states $= \mathbb{N}$, not irreducible, not recurrent

Infinitesimal generator: $Q = (q_{ij})$ with $q_{i(i+1)} = \lambda$, $q_{ii} = -\lambda$

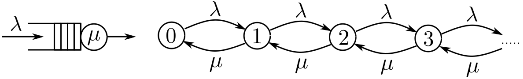
Invariant distribution: $\pi Q = 0$ has no proba distribution as solution

Remark: particular case of pure birth process

⚠ in the transition graph of Q, loops (q_{ii}) are usually not pictured

Regular jump HMC: examples

Example 3: M/M/1 Queue with exponential $Exp(\mu)$ i.i.d. service times and exponential $Exp(\lambda)$ i.i.d. interarrival times, $\lambda, \mu > 0$



Structure: states = $\mathbb{N}$, irreducible

Infinitesimal generator: $Q = (q_{ij})$ with $q_{i(i+1)} = \lambda$, $q_{i(i-1)} = \mu$

Invariant distribution:

$\pi Q = 0 \iff$ input flow = output flow everywhere

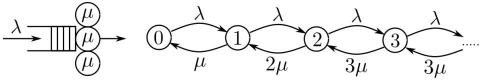
Observe input/output from $\{0, \ldots, n\}$: $\pi_n \lambda = \pi_{n+1} \mu$

Invariant non null measure: $\pi_n = \frac{\lambda^n}{\mu^n} \pi_0$, $\sum_n \pi_n < \infty$ iff $\frac{\lambda}{\mu} < 1$

Vocabulary: *at the stationary regime* = assume that starting distribution is the invariant distribution (if it exists and unique)

Regular jump HMC: examples

Example 4: M/M/3 Queue = like M/M/1 but with 3 indep servers (same service)



Structure: states = $\mathbb{N}$, irreducible

Invariant distribution: use the same kind of calculations